Permutation & Combination

INTRODUCTION Section - 1

1.1 Definition of Factorial

Factorial : The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\lfloor n$.

i.e. $n! = n(n-1)(n-2)\ldots\ldots 3 \times 2 \times 1$

Thus, $4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

1.2 Properties of factorial

(i) $n!$ is defined for positive integers only.

(ii) Factorials of proper fractions or integers are not defined. Factorial n is defined only for whole numbers.

(iii) $n! = 1 \times 2 \times 3 \times = \ldots\ldots \times (n-1) \times n$

$= [1 \times 2 \times 3 \times = \ldots\ldots \times (n-1)]\, n = (n-1)!\, n$

Thus, $n! = n\,(n-1)!$

(iv) $0! = 1$ (by definition)

(v) If two factorials, *i.e.*, $x!$ and $y!$ are equal, then $x = y$ or $x = 0$, $y = 1$ or $x = 1$, $y = 0$.

***Illustrating the Concepts* :**

If $\dfrac{n!}{2!\,(n-2)!}$ *and* $\dfrac{n!}{4!\,(n-4)!}$ *are in the ratio* 2 : 1, *then find the value of n* ?

$$\frac{n!}{2!\,(n-2)!} = \frac{n(n-1)(n-2)\ldots\ldots 3 \times 2 \times 1}{2!\,(n-2)!}$$

$$= \frac{n(n-1)\{(n-2)(n-3)\ldots\ldots 3 \times 2 \times 1\}}{2!\,(n-2)!}$$

$$= \frac{n(n-1)}{2!}\,\frac{(n-2)!}{(n-2)!} = \frac{n(n-1)}{2!}$$

Similarly,

$$\frac{n!}{4!\,(n-4)!} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{4!(n-4)!}$$

$$= \frac{n(n-1)(n-2)(n-3)}{4!}$$

Given : $\dfrac{n!}{2!\,(n-2)!} : \dfrac{n!}{4!(n-4)!} = 2 : 1$

$$\Rightarrow \quad \frac{\dfrac{n(n-1)}{2}}{\dfrac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1}} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{12}{(n-2)(n-3)} = \frac{2}{1}$$

$\Rightarrow \quad (n-2)(n-3) = 6$

$\Rightarrow \quad n^2 - 5n = 0 \qquad n = 0, 5$

But, for $n = 0$, $(n-2)!$ and $(n-4)!$ are not meaningful.

So, $n = 5$.

Illustration - 1 *The value of* $\frac{(2n)!}{n!}$ *is equal to :*

(A) $\{1.\ 3.\ 5.\ .\ .\ .\ .\ .(2n-1)\}2^n$ **(B)** $\{1.\ 3.\ 5.\ .\ .\ .\ .\ .(2n-1)\}\ 2^n\ n!$

(C) $\{1.\ 3.\ 5.\ .\ .\ .\ .\ .(2n+1)\}2^n$ **(D)** None of these

SOLUTION : (A)

Using the definition of factorials,

$$\frac{(2n)!}{n!} = \frac{1.2.3\ldots\ldots.(2n-1)(2n)}{n!}$$

Separating odd and even terms in numerator, we get :

$$= \frac{\{1.3.5\ldots\ldots(2n-1)\}\{2.4.6\ldots.2n\}}{n!} = \frac{\{1\cdot 3\cdot 5\ldots.(2n-1)\}\,2^n\ n!}{n!}$$

$$= \{1.3.5\ldots\ldots.(2n-1)\}2^n$$

1.3 Exponent of prime *p* in *n*!

Let p be a prime number and n be a positive integer. Then, the last integer amongst 1, 2, 3, ... , $(n-1)$, n which is divisible by p is $\left[\frac{n}{p}\right]p$, where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to n/p.

$E_p(n!)$ denote the exponent of p in the positive integer n. Then,

$$E_p(n!) = E_p(1.2.3\ldots\ldots(n-1)\,n)$$

$$= E_p\left(p.2p.3p\ldots\ldots\left[\frac{n}{p}\right]p\right)$$

[As Remaining integers between 1 and n are not divisible by p]

$$= \left[\frac{n}{p}\right] + E_p\left(1.2.3\ldots..\left[\frac{n}{p}\right]\right)$$

Now, the last integer amongst, 1, 2, 3, ..., $\left[\frac{n}{p}\right]$ which is divisible by p is :

$$\left[\frac{n/p}{p}\right]p = \left[\frac{n}{p^2}\right]p$$

$$\therefore \quad E_p(n!) = \left[\frac{n}{p}\right] + E_p\left(p.2p\ldots\ldots\left[\frac{n}{p^2}\right]p\right)$$

[As Remaining integers between 1 and $[n/p]$ are not divisible by p]

$$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1.2.3\ldots..\left[\frac{n}{p^2}\right]\right)$$

Continuing in the same manner, we get :

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots.. + \left[\frac{n}{p^t}\right]$$

[where t is the largest positive integer such that $p^t \le n \le p^{t+1}$]

Note : This result is not valid for composite numbers.

For Example :

This result cannot be used to find the exponent of 6 in n ! as 6 is composite of 2 and 3.

***Illustrating the Concepts* :**

Find the exponent of 2 *in* 50! ?

$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{2^2}\right] + \left[\frac{50}{2^3}\right] + \left[\frac{50}{2^4}\right] + \left[\frac{50}{2^5}\right] = 25 + 12 + 6 + 3 + 1 = 47$$

Illustration - 2 *The number of zeroes in* 100! *are* :

(A) 20 **(B)** 24 **(C)** 97 **(D)** 28

SOLUTION : (B)

We know, $10 = 5 \times 2$

So, to form one 10, we need one 2 and one 5.

Number of 10's will be same as $min\,\{E_2(100!), E_5(100!)\}$

$$E_2(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right]\left[\frac{100}{2^6}\right] = 50 + 25 + 12 + 6 + 3 + 1 = 97$$

$$E_5(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

$$E_{10}(100!) = min\,\{97, 24\} = 24$$

FUNDAMENTAL PRINCIPLE OF COUNTING — Section - 2

2.1. Addition Principle

If a work can be done in m different ways and another work which is independent of first can be done in n different ways, then either of the two operations can be performed in $(m + n)$ ways.

***Illustrating the Concepts* :**

There are 15 *gates to enter a city from north and* 10 *gates to enter the city from east. In how many ways a person can enter the city ?*

Number of ways to enter the city from north = 15

Number of ways to enter the city from east = 10

A person can enter the city from north or from east.

Hence, the number of ways to enter the city = 15 + 10 = 25